

Numerical values of the growth rates of power-free languages

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Abstract

We present upper and two-sided bounds of the exponential growth rate for a wide range of power-free languages. All bounds are obtained with the use of algorithms previously developed by the author.

This is not a research paper but rather an appendix to several research papers by the author, see [1–5]. We put together the best currently known numerical bounds on the growth rate of power-free languages over the alphabets with 2 to 15 letters. Some of these bounds were already published in the mentioned papers, some other are new. All upper bounds are obtained by the algorithm announced in [1] and described in full extent in [4]. For all β -power-free languages with $\beta \geq 2$, the obtained upper bounds were converted to the two-sided ones using the results of [2].

Recall that the (exponential) growth rate $\alpha(L)$ of a factorial language L is defined by $\alpha(L) = \lim_{n \rightarrow \infty} (C_L(n))^{1/n}$, where $C_L(n)$ is the number of words of length n in L . We write $\alpha(k, \beta)$ for the growth rate of the k -ary β -power-free language. All upper bounds are obtained as growth rates of some regular languages. Such a “regular approximation” of the target power-free language consists of all words avoiding all forbidden powers having the periods bounded from above by some constant m . (For example, take the cube-free language over the alphabet $\{a, b\}$ and $m = 1$. The corresponding regular approximation is the language of all binary words having no factors aaa and bbb . Its growth rate is the golden ratio; this is the simplest nontrivial upper bound for $\alpha(2, 3)$.) Lower bounds are obtained from the upper ones by a very simple (almost constant-time) computation.

We present a separate table for binary languages; three other tables are devoted to the languages over different alphabets, avoiding “big”, “average”, and “small” powers, respectively. All bounds are rounded off to 7 decimal places. For β -power-free languages with $\beta < 2$, we give only upper bounds. We also present our estimation of the actual value of the growth rate of each of these languages; such an estimation is obtained by extrapolation from a series of successive upper bounds.

The growth of some particular power-free languages was extensively studied by different authors. The binary 2^+ -power-free language is known to have polynomial growth, so its growth rate equals 1. The same is true for binary $(7/3)$ -power-free language. For the growth rate of the binary cube-free language we present the upper bound 1.4575772869240. The estimated exact value of this growth rate is

about 1.4575772869237 ($3 \cdot 10^{-13}$ from the upper bound). The best proved lower bound is 1.4575732 ($4 \cdot 10^{-6}$ from the upper bound). Finally, for the growth rate of the ternary square-free language our best upper bound is 1.301761876 , the estimated exact value is about 1.30176183 , and the best lower bound is 1.3017597 .

Table 1 shows the behaviour of the growth rate of binary power-free languages. All points in which the function $\alpha(2, \beta)$ jumps by at least 0.001 are included in this table. The sum of jumps in these points is about 0.98 . Thus, the behaviour of $\alpha(2, \beta)$ can be seen in details. For $\beta \geq 7/2$, we use lower bounds with “double precision”. The validity of such bounds for the case $\beta \geq 4$ was proved in [2]; the argument of [2] can be strengthened to lower the bound on β to $7/2$.

Table 1: Binary power-free languages. *If a cell contains only one number, this number is the exact growth rate of the binary β -free language, rounded off to 7 decimal places. Otherwise, the cell contains the lower and the upper bounds to such a growth rate; these bounds are also rounded off to 7 decimal places. The integer in the same cell is the number m of the corresponding approximation. The amount of jump of the growth rate at the point β is shown in the last column.*

β	β -free	β^+ -free	jump at β
$7/3$	1.0000000	65 1.2206318–1.2206448	0.2206
$17/7$	64 1.2222235–1.2222380	63 1.2287081–1.2287205	0.0065
$5/2$	62 1.2294871–1.2295017	44 1.3662971–1.3663011	0.1368
$18/7$	43 1.3669547–1.3669601	43 1.3692782–1.3692832	0.0023
$13/5$	43 1.3693912–1.3693962	42 1.3760821–1.3760876	0.0067
$8/3$	42 1.3762649–1.3762704	37 1.4508577–1.4508611	0.0746
$14/5$	36 1.4522648–1.4522680	36 1.4552314–1.4552358	0.0030
$17/6$	36 1.4552552–1.4552596	36 1.4567773–1.4567815	0.0015
3	36 1.4575732–1.4575773	24 1.7951246–1.7951264	0.3375
$13/4$	24 1.7957598–1.7957616	24 1.7972871–1.7972888	0.0015
$10/3$	24 1.7973088–1.7973105	24 1.8029861–1.8029877	0.0057
$7/2$	1.8032409	1.8172665	0.0140
$11/3$	1.8174176	1.8204960	0.0031
4	1.8211000	1.9208015	0.0997
$9/2$	1.9214442	1.9241348	0.0027
5	1.9244437	1.9646285	0.0402
6	1.9653118	1.9832942	0.0180
7	1.9834409	1.9918972	0.0085
8	1.9919310	1.9960151	0.0041
9	1.9960232	1.9980255	0.0020

Our Table 2 contains the growth rates of the β -power-free languages with $\beta \geq 2$. Some obvious laws of the behaviour of the function $\alpha(k, \beta)$ can be seen in this table; see [5] for details.

In Table 3 we study the powers that are less than 2 but still relatively big. Since the values $\alpha(k, \frac{3}{2}^+)$ and $\alpha(k, 2)$ are quite close to each other for any $k \geq 4$, we do not consider the powers β such that $\frac{3}{2}^+ < \beta < 2$. The laws observed in Table 3 are also explained in [5].

The results given in Table 4 concern about small power-free languages. It is known from Dejean’s conjecture (finally proved in 2009) that the k -ary β -power-

Table 2: Avoiding big exponents: $\beta \geq 2$. If a cell contains one number, this number is the exact growth rate of the k -ary β -free language, rounded off to 7 decimal places. Two numbers in a cell are the lower and the upper bounds of such a growth rate; these bounds are also rounded off to 7 decimal places.

$k \backslash \beta$	2	2 ⁺	3	3 ⁺	4	4 ⁺
	1.3017597–	2.6058789–	2.7015614–	2.9119240–		
3	1.3017619	2.6058791	2.7015616	2.9119242	2.9172846	2.9737546
4	2.6215080	3.7284944	3.7789513	3.9487867	3.9507588	3.9879972
5	3.7325386	4.7898507	4.8220672	4.9662411	4.9671478	4.9935251
6	4.7914069	5.8277328	5.8503616	5.9760100	5.9764861	5.9961170
7	5.8284661	6.8537250	6.8705878	6.9820558	6.9823298	6.9974912
8	6.8541173	7.8727609	7.8858522	7.9860649	7.9862337	7.9982866
9	7.8729902	8.8873424	8.8978188	8.9888625	8.9889721	8.9987785
10	8.8874856	9.8988872	9.9074705	9.9908932	9.9909674	9.9990989
11	9.8989813	10.9082635	10.9154294	10.9924142	10.9924662	10.9993163
12	10.9083279	11.9160348	11.9221106	11.9935831	11.9936207	11.9994691
13	11.9160804	12.9225835	12.9278022	12.9945010	12.9945288	12.9995796
14	12.9226167	13.9281788	13.9327109	13.9952350	13.9952560	13.9996615
15	13.9282035	14.9330157	14.9369892	14.9958311	14.9958473	14.9997234

free language is infinite iff $\beta \geq (7/4)^+$ for $k = 3$, $\beta \geq (7/5)^+$ for $k = 4$, and $\beta \geq \frac{k}{k-1}^+$ for all other k . To make Table 4 readable, we give the information about the minimal power-free languages over the ternary and quaternary alphabets in the column headed by $\frac{k}{k-1}^+$. The conjectures about the behaviour of the function $\alpha(k, \beta)$ for the case of small β are given in [3, 5].

References

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Table 3: Avoiding average exponents: $5/4 \leq \beta \leq (3/2)^+$. First row of each cell contains the best obtained upper bound for the growth rate of the k -ary β -free language, rounded off to 7 decimal places, and the number m of the corresponding approximation. The bottom number in the cell is our approximation to the growth rate of the k -ary β -free language obtained by the extrapolation of the series of upper bounds.

$k \backslash \beta$	$5/4$	$(5/4)^+$	$4/3$	$(4/3)^+$	$3/2$	$(3/2)^+$
4	0	0	0	0	154 1.0968025 1.09679	21 2.2805723 2.28052
5	0	112 1.1580040 1.1577	109 1.1646054 1.1645(6)	24 2.2489630 2.2485	22 2.4024448 2.40242(4)	16 3.4928277 3.492800
6	91 1.2289301 1.2288	25 2.2781859 2.277(3)	24 2.3765882 2.3763(9)	17 3.4008105 3.4003(9)	17 3.5405292 3.540514	14 4.6034033 4.603386
7	26 2.3267948 2.3265	19 3.3737870 3.3729	19 3.4660659 3.4658(8)	15 4.5026780 4.50224	15 4.6275095 4.627498	13 5.6727100 5.672703
8	20 3.4104475 3.4101	16 4.4532917 4.4515	16 4.5400020 4.53976	14 5.5749102 5.57467	14 5.6867745 5.686769	13 6.7206899 6.720687
9	18 4.4785124 4.4778	16 5.5160143 5.5150(2)	15 5.5982898 5.59814	14 6.6287238 6.62861	14 6.7296776 6.729676	13 7.7560582 7.756057
10	17 5.5342518 5.5336(4)	15 6.5670585 6.5663	15 6.6443657 6.64428	14 7.6703573 7.67029(4)	14 7.7621761 7.762175	12 8.7832930 8.783291
11	17 6.5802111 6.57983	15 7.6086719 7.6081(7)	15 7.6813567 7.68131	14 8.7035224 8.70348(6)	14 8.7876552 8.787655	12 9.8049486 9.804948
12	17 7.6185200 7.61827	15 8.6432160 8.6428(8)	15 8.7115615 8.71153(5)	14 9.7305688 9.730547	13 9.8081756 9.808175	12 10.8226038 10.822603
13	17 8.6507655 8.65058(9)	15 9.6722706 9.67204	15 9.7366286 9.736612	14 10.7530530 10.753039	13 10.8250601 10.825060	12 11.8372861 11.837286
14	17 9.6781738 9.67804(8)	15 10.6970094 10.69685	15 10.7577374 10.757727	14 11.7720444 11.772035	13 11.8392005 11.839200	12 12.8496956 12.849695
15	17 10.7017089 10.701616	15 11.7183082 11.71819	15 11.7757426 11.775735	14 12.7883022 12.788296	13 12.8512175 12.851217	12 13.8603271 13.860327

Table 4: Avoiding small exponents: $\beta \leq (k-3)/(k-4)$. First row of each cell contains the best obtained upper bound for the growth rate of the k -ary β -free language, rounded off to 7 decimal places, and the number m of the corresponding approximation. The bottom number in the cell is our approximation to the growth rate of the k -ary β -free language obtained by the extrapolation of the series of upper bounds.

$k \backslash \beta$	$\frac{k-1}{k-1}^+$	$\frac{k-1}{k-2}$	$\frac{k-1}{k-2}^+$	$\frac{k-2}{k-3}$	$\frac{k-2}{k-3}^+$	$\frac{k-3}{k-4}$
3*	66 1.2456093 1.245608					
4*	208 1.0695061 1.0694	156 1.0968016 1.09679	21 2.2805723 2.28052			
5	115 1.1579787 1.1577	112 1.1645978 1.1645(6)	24 2.2489630 2.2485	22 2.4024448 2.40242(4)	16 3.4928277 3.492800	
6	95 1.2247121 1.2246	92 1.2289256 1.2288	25 2.2781859 2.277(3)	24 2.3765882 2.3763(9)	17 3.4008105 3.4003(9)	17 3.5405292 3.540514
7	100 1.2369024 1.2368(6)	99 1.2373991 1.2373(7)	27 2.299738 2.2990	26 2.3267948 2.3265(7)	19 3.3737870 3.3729	19 3.4660659 3.4658(8)
8	109 1.2348427 1.23483	100 1.2349430 1.23494	29 2.3132472 2.312(7)	29 2.3285295 2.327(0)	20 3.3639604 3.361(7)	20 3.4104475 3.4101
9	107 1.2466776 1.24667	100 1.2467443 1.24674	31 2.3194735 2.319(0)	31 2.3261560 2.326(0)	22 3.3639868 3.362	22 3.3904233 3.390
10	109 1.2393075 1.239307	100 1.2393375 1.239337	31 2.3235082 2.322(9)	32 2.3259292 2.325(5)	23 3.3669859 3.364	23 3.3796610 3.377
11	110 1.2426060 1.242606	92 1.2426389 1.242638(7)	31 2.3273701 2.324	31 2.3273701 2.325	25 3.3693164 3.367	25 3.3759580 3.374
12	108 1.2428777 1.242877(5)	96 1.2428801 1.242880	33 2.3289650 2.325	33 2.3289650 2.326	26 3.3714325 3.370	27 3.3745058 3.374
13	104 1.2408703 1.240870(1)	96 1.2408753 1.240875	36 2.3294446 2.326	36 2.3294446 2.326	26 3.3760035 3.372	26 3.3762406 3.37(5)
14	105 1.2427746 1.242774(2)	98 1.2427761 1.242776	38 2.3297907 2.326	38 2.3297907 2.326	26 3.3863857 3.374	26 3.3864742 3.37(6)
15	105 1.2418324 1.241832	90 1.2418340 1.241833	41 2.3299169 2.327	41 2.3299169 2.327	27 3.3942863 3.375	27 3.3943189 3.37(7)